# On a transformation between hierarchies of integrable equations

Metin Gürses[†] and Kostyantyn Zheltukhin[‡]

[†]Department of Mathematics, Faculty of Sciences

Bilkent University, 06800 Ankara, Turkey

e-mail gurses@fen.bilkent.edu.tr

[‡]Department of Mathematics, Faculty of Sciences

Middle East Technical University 06531 Ankara, Turkey

e-mail zheltukh@metu.edu.tr

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**Abstract**

A transforation between a hierarchy of integrable equations arising from the standard $R$-matrix construction on the algebra of differential operators and a hierarchy of integrable equations arising from a deformation of the standard $R$-matrix is given.

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In a recent paper [1] a new hierarchy of integrable equations has been constructed through the deformation of a standard $R$-matrix on the algebra of pseudo-differential operators. We give a transformation between the hierarchy constructed in [1] and a hierarchy obtained through a standard $R$-matrix. The transformation is between corresponding vector fields (i.e. symmetries).

Let $\mathfrak{g}$ be the Lie algebra of pseudo-differential operators

$$\mathfrak{g} = \left\{ \sum_{i \in \mathbf{Z}} u_i(x) D^i \right\} \tag{1}$$

with the commutator $[L_1, L_2] = L_1 L_2 - L_2 L_1$. The algebra $\mathfrak{g}$ can be decomposed into Lie subalgebras $\mathfrak{g}_{\geq k} = \left\{ \sum_{i \geq k} u_i(x) D^i \right\}$ and , $\mathfrak{g}_{i<k} = \left\{ \sum_{i<k} u_i(x) D^i \right\}$ where $k = 0, 1, 2$ (only for such $k$ one has Lie subalgebras). The standard $R$-matrix is given by $R_k = \frac{1}{2}(P_{\geq k} - P_{<k})$, where $P_{\geq k}$ and $P_{<k}$ are projection operators on $\mathfrak{g}_{\geq k}$ and $\mathfrak{g}_{<k}$, respectively. The Lax hierarchy is

$$L_{t_n} = [R(L^n), L] = [(L^n)_{\geq k}, L], \qquad L \in \mathfrak{g}, \qquad n = 1, 2 \ldots . \tag{2}$$

The above equations involves infinitely many fields. To have a consistent closed equations with a finite number of fields we restrict the Lax operators as follows

$$k = 0 \quad L_0 = D^N + u_{N-2} D^{N-2} + \cdots + u_1 D + u_0 \tag{3}$$

$$k = 1 \quad L_1 = D^N + u_{N-1} D^{N-1} + \cdots + u_0 + D^{-1} u_{-1} \tag{4}$$

$$k = 2 \quad L_2 = u_N D^N + u_{N-1} D^{N-1} + \cdots + D^{-1} u_{-1}. + D^{-2} u_{-2} \tag{5}$$

See [2] for more details on the $R$-matrix formalism.

Recently in [1] the deformations of the above $R$-matrices were introduced. Most of the introduced deformed $R$-matrices do not lead to the new hierarchies. A new hierarchy is obtained through a deformation of $R$-matrix $R_1 = \frac{1}{2}(P_{\geq 1} - P_{<1})$. Let $P_{=i}(L) = (L)_{=i}$ denotes coefficient of $D^i$ in the expansion of $L \in \mathfrak{g}$. Then the deformed $R$-matrix is

$$\tilde{R} = \frac{1}{2}(P_{\geq 1} - P_{<1}) + \varepsilon P_{=0}(\cdot)D, \tag{6}$$

where $\varepsilon$ is a deformation parameter. The hierarchy is

$$L_{t_n} = [\tilde{R}(L^n), L], \qquad L \in \mathfrak{g}, \qquad n = 1, 2 \ldots . \tag{7}$$

The above equations involves infinitely many fields, to have a consistent closed equation with finite number of fields we restrict the Lax operator as $\tilde{L} = u_N D^N + u_{N-1} D^{N-1} + \cdots + u_0 + D^{-1} u_{-1}$. Then the new hierarchy is

$$\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 1} + \epsilon (\tilde{L}^n)_{=0} D, \tilde{L}], \qquad n = 1, 2 \ldots, \tag{8}$$

note that $\tilde{L} = L_2|_{u_{-2}=0}$. See [1] for more details.

In this work we shall show that the new hierarchy (8) is related to the hierarchy corresponding to $R$-matrix $R_2$ with reduced Lax operator $\tilde{L} = L_2|_{u_{-2}=0}$. So we relate hierarchy (8) to the hierarchy

$$\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 2}, \tilde{L}], \qquad n = 1, 2 \ldots . \tag{9}$$

We note that both hierarchies have the same Lax operator.

The construction of the transformation is based on expressing $(\tilde{L}^n)_{=1}$ and $(\tilde{L}^n)_{=0}$ in terms of coefficients of $[(\tilde{L}^n)_{\geq 2}, \tilde{L}]$, for $n \in \mathbf{N}$.

**Proposition 1**. *Let* $\tilde{L} = L_2|_{u_{-2}=0}$, *then*

$$([(\tilde{L}^n)_{\geq 2}, \tilde{L}])_{=N} = -([(\tilde{L}^n)_{=1} D, \tilde{L}])_{=N}, \tag{10}$$

$$([(\tilde{L}^n)_{\geq 1}, \tilde{L}])_{=N-1} = -([(\tilde{L}^n)_{=0}, \tilde{L}])_{=N-1}. \tag{11}$$

*for all* $N$ *(*$N$ *is order of operator* $\tilde{L}$*).*

**Proof**. Comparing powers of $D$ on the right and left hand side of the equality

$$[(\tilde{L}^n)_{\geq 1}, \tilde{L}] = -[(\tilde{L}^n)_{<1}, \tilde{L}], \tag{12}$$

we have

$$([(\tilde{L}^n)_{\geq 1}, \tilde{L}])_{=N} = 0. \tag{13}$$

Then

$$([(\tilde{L}^n)_{\geq 2}, \tilde{L}])_{=N} = -([(\tilde{L}^n)_{=1} D, \tilde{L}])_{=N}. \tag{14}$$

In the same way, comparing powers of $D$ on the right and left hand side of the equality

$$[(\tilde{L}^n)_{\geq 0}, \tilde{L}] = -[(\tilde{L}^n)_{<0}, \tilde{L}] \tag{15}$$

we have

$$([(\tilde{L}^n)_{\geq 0}, \tilde{L}])_{=N-1} = 0. \tag{16}$$

So,

$$([(\tilde{L}^n)_{\geq 1}, \tilde{L}])_{=N-1} = -([(\tilde{L}^n)_{=0}, \tilde{L}])_{=N-1}. \tag{17}$$

The above equalities (10) and (11) allows us to express $(\tilde{L}^n)_{=1}$ and $(\tilde{L}^n)_{=0}$ in terms of coefficients of $[(\tilde{L}^n)_{\geq 2}, \tilde{L}]$ for all $N$. Let us give an example for $N = 1$.

**Proposition 2**. *Consider the Lax operator $\tilde{L} = uD + v + D^{-1}w$. Let*

$$[(\tilde{L}^n)_{\geq 2}, \tilde{L}] = f_n D + g_n + D^{-1}h_n, \tag{18}$$

*which gives the hierarchy (9) with the standard R-matrix and*

$$[(\tilde{L}^n)_{\geq 1} + (\tilde{L}^n)_{=0}D, \tilde{L}] = p_n D + q_n + D^{-1}r_n, \tag{19}$$

*which gives the hierarchy (8) with the deformed R-matrix, $n = 1, 2 \ldots$. The coefficients $f_n$, $g_n$, $h_n, p_n$, $q_n$, $r_n$ are functions of $u$, $v$, $w$ and their derivatives. Then*

$$(p_n,\, q_n,\, r_n)^T = \mathcal{T}\,(f_n,\, g_n,\, h_n)^T$$

*where*

$$\mathcal{T} = \begin{pmatrix} \varepsilon u_x D^{-1} v_x D^{-1} u^{-2} - \varepsilon u v_x D^{-1} u^{-2} & \varepsilon u_x D^{-1} u^{-1} - \varepsilon & 0 \\ \\ u v_x D^{-1} u^{-2} + \varepsilon v_x D^{-1} v_x D^{-1} u^{-2} & 1 + \varepsilon v_x D^{-1} u^{-1} & 0 \\ \\ ((uw)_x + \varepsilon w v_x) D^{-1} u^{-2} & \varepsilon w u^{-1} + \varepsilon w_x D^{-1} u^{-1} & 1 \\ +\varepsilon w_x D^{-1} v_x D^{-1} u^{-2} + w u^{-1} & & \end{pmatrix} \tag{20}$$

**Proof.** Let $(\tilde{L}^n)_{=1} = A_n$ and $(\tilde{L}^n)_{=0} = B_n$. The equality (10) implies that

$$f_n = -([A_n D, \tilde{L}])_{=1}, 0 \tag{21}$$

hence, we can find

$$A_n = u D^{-1} u^{-2} f_n. \tag{22}$$

Using the equality (11) we have

$$g_n + ([A_n D, \tilde{L}])_{=0} = -([B_n, \tilde{L}])_{=0}, \tag{23}$$

hence, we can find

$$B_n = D^{-1}(u^{-1}g_n + v_x D^{-1}u^{-2}f_n). \tag{24}$$

From the equality

$$[(\tilde{L}^n)_{\geq 1} + \varepsilon(\tilde{L}^n)_{=0}D, \tilde{L}] = [(\tilde{L}^n)_{\geq 2}, \tilde{L}] + [(A_n + \varepsilon B_n)D, \tilde{L}] \tag{25}$$

we can find the transformation between the vector fields

$$\begin{aligned} p_n &= u_x\varepsilon B_n - u\varepsilon B_{n,x} \\ q_n &= g_n + v_x(A_n + \varepsilon B_n) \\ r_n &= h_n + (w(A_n + \varepsilon B_n))_x \end{aligned} \tag{26}$$

where $A_n$ and $B_n$ are given by (22) and (24) respectively. Thus we obtain the transformation operator $\mathcal{T}$ in (20).

If we apply operator $\mathcal{T}$ to the simple symmetry $(u_x, v_x, w_x)^T$ we obtain $(0,0,0)^T$. Applying the operator $\mathcal{T}$ to $(0,0,0)^T$ we get

$$\begin{pmatrix} p_1 \\ q_1 \\ r_1 \end{pmatrix} = \begin{pmatrix} \varepsilon(vu_x - uv_x + u_x) \\ uv_x + \varepsilon(vv_x + v_x) \\ (uw)_x + \varepsilon(vw)_x + \varepsilon w_x \end{pmatrix}. \tag{27}$$

This is the deformed system (8) for $n = 1$ (with the inclusion of the symmetry $(u_x, v_x, w_x)^T$), [1]. If we take symmetry of the hierarchy (9) corresponding to $n = 2$ (this is the reduced system [2], [3])

$$\begin{pmatrix} f_2 \\ g_2 \\ h_2 \end{pmatrix} = \begin{pmatrix} u^2u_{xx} + 2u^2v_x \\ u^2v_{xx} + 2u(uw)_x \\ -(u^2w)_{xx} \end{pmatrix} \tag{28}$$

and apply the operator $\mathcal{T}$ to this symmetry we obtain a second symmetry of the hierarchy (8)

$$\begin{pmatrix} p_2 \\ q_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} \varepsilon u_x v^2 - 2\varepsilon u v v_x - 2\varepsilon u^2 w_x - \varepsilon u^2 v_{xx} \\ \\ 2uu_x w + 2uvv_x + 2u^2 w_x + uu_x v_x + u^2 v_{xx} + \varepsilon v^2 v_x + 2\varepsilon u v_x w + \varepsilon u v_x^2 \\ \\ 2u_x vw + 2uv_x w + 2uvw_x - u_x^2 w - 3uu_x w_x - uu_{xx} w - u^2 w_{xx} + \\ 2\varepsilon u_x w^2 + 2\varepsilon v v_x w + \varepsilon u_x v_x w + \varepsilon v^2 w_x + 4\varepsilon u w w_x + \varepsilon u v_x w_x + \varepsilon u v_{xx} w \end{pmatrix}. \tag{29}$$

**Remark.** In the example above we have constructed the transformation $\mathcal{T}$ for hierarchies with Lax operator of order one. In the same way we can construct the transformation between hierarchies with Lax operator of any order $N$. The operator $\mathcal{T}$ is not a recursion operator. It maps the symmetries of one system of evolution equations to symmetries of another system of evolution equations.

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